Solutions of the Boussinesq Equations through Bifurcation Method

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Abstract:
The article established the nonlinear theory to find the solution using a new notion of bifurcation known as attractor bifurcation. It determined the bifurcation and stability of the solutions of the Boussinesq equations as well as the onset of the Rayleigh-Benard convection. In this article we considered the theory that comprises the succeeding perspectives. The study initially deal with the problem that bifurcates from the trivial solution an attractor A_R while the Rayleigh number R intersects the first critical Rayleigh number R_C for all physically boundary conditions, despite the multiplicity of the eigenvalue R_C for the linear problem. Hereafter, secondly, the study measured the bifurcated attractor A_R as asymptotically stable. Finally, the bifurcated solutions are also structurally stable when the spatial dimension is two, and are classified as a bifurcated solution as well. Furthermore, the technical method explained here provides a means, which can be adopted for many differen.

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1.Introduction

There are general concerned and a fully understanding in the concept of matter and its compositions. It leads with the fully understood of something flows from a hot bodies/objects to a cold bodies/objects . The phenomenon of flows is known as “heat.” During the eighteenth and the early nineteenth centuries many scientific approach revealed that all bodies consists of an invisible fluid within it known to be caloric[1]. Hence this caloric has a variety of properties some of which proved to be inconsistent with nature, for instance it has weight and cannot be created nor destroyed. However, it flows from hot bodies to the cold bodies and this was considered as among the most important feature of it. Therefore, it is important to consider heat as a valuable aspect of live[2]

In a nutshell thermal convection refers to a specific type of convection phenomena where temperature differences drive a fluid flow. More precisely temperature variations induce an unstable fluid stratification which cause the transition of the fluid from a state of rest to a state of motion [3]. The fluid flow may undergo much successive instability, which reduce the spatial coherence and the level of predictability of the details of movement progressively. In this case, the flow is called turbulent. Few examples of (turbulent) thermal convection are air circulation, solar granulation, oceanic currents and convective flows in the earth's mantle and stars. Transport properties of turbulent convective flow are the object of interest and investigation in many field ranging from physical sciences like geophysics, astrophysics, meteorology, and oceanography to engineering and industrial applications [4].

A fluid heated from the bottom and cooled at the surface in a cylindrical container will cause convection if the temperature difference (ΔT) between the surface and the base plates is at least has a critical temperature difference (ΔTc). The phenomenon above is called Rayleigh-Benard convection[5], or in short form as RBC. However, convection does not occur in the fluid when $\Delta T < \Delta Tc$, due to viscous and thermal dissipation and will settled in what is called the "conducting" or "uniform" solution. Therefore, whenever ΔT large enough, convection is will occur as the thermal driving force is significant enough to overcome the dissipative effects of thermal conduction and viscosity [6]. Convection will only happen when the dimensionless control parameter, the Rayleigh number

$$\mathrm{Ra} = \frac{\propto gH^3}{vk}\Delta T \tag{1.1}$$

Attains a critical value Rac, the α is the thermal-expansion coefficient, g is the acceleration due to gravity, H represent the fluid thickness layer[7] , where ΔT stand for the temperature ladder, κ is for the thermal diffusivity with the ν as the kinematic viscosity. In standard the Rayleigh number shows the characterizes ratio of the undermining buoyancy force ραgΔT in respect to steadying dissipative force νκρ/H3. It can be stated that

$$\in = \frac{\mathrm{Ra} - \mathrm{Rac}}{\mathrm{Rac}} \tag{1.2}$$

In order to normalize the degree over a threshold; a certain Rayleigh number is for meant for a specific aspect ratio. The dimensionless Prandtl number

$$\mathrm{Pr} = \frac{v}{k} \tag{1.3}$$

Gives the properties of the fluid including the dimensionless aspect ratio

$$\Gamma \equiv \frac{D}{H} \tag{1.4}$$

Where D is the diameter and H is the depth of the cylinder, characterizes the geometry.

It is perfect to identify and noted that a complete nonlinear bifurcation and stability theory for this problem must at any rate contain the following aspects:

a) The bifurcation theorem while the Rayleigh number bisected the initial critical number for all the physically boundary conditions,

b) The asymptotic stability of bifurcated solutions, and lastly

c) The pattern or structure and their stability and transitions within the physical space.

The leading difficulties concerning such a complete theory are two-fold. Initially is due to the high nonlinearity of the problem as in other fluid problems, also secondly is due to the lack of an approach to handle bifurcation and stability at the eigenvalue of the linear problem has even multiplicity[8].

The main aim of this research is to reduce the bifurcation problems to the centre manifold together with an S_1 attractor bifurcation theorem and structural stability theorem for 2D incompressible flows to achieve the following objectives:

1. To classify the solutions in the bifurcated attractor A_R.
2. To study the structure and its transition of the solution of the Benard problem in the physical space.
3. To study the dynamic bifurcation and the structural stability of the bifurcated solutions of the 2-D Boussinesq equations related to the Rayleigh-Benard convection.

2. Methodology

In this article, we are interested in deriving mathematically rigorous bounds for the heat transport when the flow is turbulent. For this purpose, consider a fluid enclosed between two rigid parallel and infinitely extended plates separated by a vertical distance h and held at different temperatures T = Tbottom and T = Ttop at height 0 and h respectively, with Tbottom > Ttop. This model of thermal convection goes under the name of Rayleigh-Benard convection[9].

In this work, the technique utilized in achieving our objectives was highlighted. the study emphasizes the main theorem in respect to attractor bifurcation states that for $m + 1 (m \geq 0)$ eigenvalues passing the imaginary axis,the control parameter need to crosses some certain critical value. In this article, we focus on obtaining mathematically rigorous bounds as regard heat transport for turbulent flow and this reason[10], using a fluid placed between 2 rigid parallel with infinitely extended plates that are separated by a vertical distance h having varying temperaturesT= Tbottom and T = Ttop at height 0 and h accordingly, with Tbottom. This type of thermal convection was placed under what is called Rayleigh-Benard convection[11].

In the first place, the study depicts that when the R (Rayleigh number). passes the initial critical value Rc, then the bifurcation occur for Boussinesq equations from the trivial solution an attractorA_R, having dimension fall between m and $m + 1$. In this instance, the R_C Initial critical Rayleigh number is illustrated to serve as the first eigenvalue in terms of the linear eigenvalue problem while $m + 1$ serve as the multiplicity of this eigenvalue Rc. In respect to known outcomes[12], the theorem on bifurcation achieved in this research is for the entire cases having multiplicity $m + 1$ of the critical eigenvalue Rc for the Benard problem base on any set of boundary conditions that are physical. when the trivial solution attains unstable as the R passes the value R_C , as the A_R does'nt possess this trivial solution[13].

Moreover, being an attractor, the bifurcated attractor A_R resulted to asymptotic stability indicating that it absorbs the entire solutions with primary set of data in the phase space outside of the stable manifold[14], in line with co-dimension, of the trivial solution.that in an ideal stability theorem should involve all physically significant perturbations and propagate the local stability of a chosen class of stable solutions, and presently this purpose are still battling with. However, fluid flows are often depended on times. However, bifurcation simulation for steady-state problems usually gives partial results to the problem, which is not adequate for finding the stability problem. Then it seems that the perfect aim of asymptotic stability preceded by the foremost bifurcation needs to be best explained by the attractor near, without the trivial state[15]. It is our major concern for adopting attractor bifurcation, and we expect to obtain the bifurcated attractor that are stable in this research will impact an ideal stability theorem. Still, the other critical aspect of a whole nonlinear theory for the Rayleigh-Benard convection is to section the pattern of the solutions immediately after the bifurcation[16]. A standard procedure to solve the problem enumerated above is the pattern stability of the solutions when considering physical space. Many types of studies have explored to achieve this outcome, and enact a theory that is systematic base on pattern stability as well as bifurcation of 2-D divergence-free vector fields[17] . This research depicts that for the two-dimensional instances, with any initial data that fall outside the stable manifold of the trivial solution, the solution of the Boussinesq equations will have the roll structure as is large enough.

In the actual sense, the mention outcomes for the Rayleigh-Benard convection are obtained by utilizing a new approach of dynamic bifurcation, termed attractor bifurcation[18] . The primary theorem that discussed the attractor bifurcation describes that as the control parameter passes a particular critical value as for $m + 1 (m \geq 0)$ eigenvalues intersecting the imaginary axis, then the system bifurcates due to a trivial steady state solution to an attractor having dimension that fall between m and $m + 1$, when the critical state is in asymptotic stable state. This emergent bifurcation theory completes the stated termed bifurcation theories. There exist some essential characteristics of attractor bifurcation[19]. Firstly, the A_R does not involve the trivial steady state, and is still stable; therefore, it is relevant in physical space. Additionally, the attractor involves a series of solutions regarding the evolution equation, containing perhaps heteroclinic orbits, steady states, periodical orbits, and homoclinic orbits[. Also, it shows a unified suggestion on dynamic bifurcation and this can be employed to various problems relating to mechanics and physics. Then,the number of eigenvalues $m + 1$ passing the imaginary axis should be an odd number, and the Hopf bifurcation is for the scenaro when $m + 1 = 2$. Although, the updated attractor bifurcation theorem achieved in this research can be adopted in the cases involve all $m \geq 0$.

Notably, the A_R, as described earlier, is stable, that is another noticeable issue for other established bifurcation theorems[20]. Therefore, we examined the asymptotic stability of the crucial state; an additional analysis necessary for the eigenvalues problems is the linearized problem. The study employs the Theorem 2 to demonstrate a technique of achieving asymptotic stability of the crucial state in problems that contain symmetric linearized equations. This theorem is superb; the asymptotic stability of the trivial solution to the Rayleigh-Benard problem is demonstrated[21]. This research approved this theorem as it helps find a solution to problems in several issues involving mathematical physics in line with symmetric linearized equations. The studies are itemized as follow. Initially, the research revisits of the Boussinesq equations and their respectful mathematical setting, likewise identifies some predetermine existence and uniqueness outcomes of the solutions. The overall work was summaries as enumerated in the next section where the primary attractor bifurcation theory form [22].the subsequent section in this research employ Theorem 2, as regard the asymptotic stability for the critical state problems in an evolution model having symmetric linearized equations. It was stated and also proved the main attractor bifurcation outcomes using the Raleigh-Benard convection[23].

3. Preliminary Outcomes (Boussinesq equations and their corresponding mathematical setting)

3.1 Boussinesq equations

The Boussinesq equations depicts a model in the large scale atmospheric which include oceanic flows that determine cold fronts as well as the jet stream.

$$\frac{\partial u}{\partial t} + (u\,.\nabla)\,u - v\Delta u + \rho_0{}^{-1}\nabla p = -gk[\,1 - \alpha\,(T - T_0)] \qquad (3.1)$$

$$\frac{\partial T}{\partial t} + (u\,.\nabla)\,T - k\Delta T = 0 \qquad (3.2)$$

$$\text{div}\, u = 0 \qquad (3.3)$$

Where v, k, α, g all represent constant values, and $u = (u_1, u_2, u_3)$ refers to the velocity field, p denote pressure function, T stand as the temperature function, T_0 is the constant at the beneath surface temperature at $x_3 = 0$ and $k = (0, 0, 1)$ then the unit vector is shown in the x-direction.

In order to turn the equations to non-dimensional, we use the following expressions:

$$x = hx',$$

$$t = h^2t'/k,$$

$$u = ku'/h,$$

$$T = \beta h\left(\frac{T'}{\sqrt{R}}\right) + T_0 - \beta hx_3',$$

$$p = \rho o k^2 p'/h^2 + po - gpo(hx_3' + \alpha\beta h^2(x_3')^2/2,$$

$$p_r = v/k$$

For this point, the Rayleigh number R is determine by equation (3.1), and $p_r = \nu/\kappa$ stand for the Prandtl number. By exonerating the primes, the equations (3.2) to (3.4) can be restructured in the format below.

$$\frac{1}{p_r}\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p\right] - \Delta u - \sqrt{R}Tk = 0 \qquad (3.4)$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \sqrt{R v_3} - \Delta T = 0, \qquad (3.5)$$

$$\operatorname{div} v = 0. \qquad (3.6)$$

Then, the non-dimensional domain for this respect is $\Omega = D \times (0,1) \subset R_3$, for which the relation $D \subset R_2$ is an open set[1]. And the coordinate system is stated as $x = (x_1, x_2, x_3) \in R^3$.

Though, the Boussinesq equations shown in equations (3.4) to (3.6) are the required equations to explore the Rayleigh B´enard problem given in this research. The initial value conditions use for their complement:

$$(u, T) = (u_0, T_0) \quad \text{at } t = 0 \qquad (3.7)$$

Hence, boundary conditions are required both at the top and the bottom as well as the lateral boundary $\partial D \times (0,1)$, and the top and bottom boundary will be $(x_3 = 0, 1)$, whichever the so-called rigid or free boundary conditions are given:

$$T = 0, u = 0 \text{ (rigid boundary)}, \qquad (3.8)$$

$$T = 0,\ u_3 = 0,\ \frac{\partial(u_1, u_2)}{\partial x_3} = 0 \text{ (free boundary)}. \qquad (3.9)$$

Mostly various combinations are utilized at the top and beneath boundary conditions in vary physical setting which includes the system of free-rigid, rigid-rigid, free-free, rigid-free and free-rigid. likewise, for the lateral boundary $\partial D \times [0,1]$, is commonly employed one of the highlight boundary conditions as in[24]:

1. The Periodic condition:

$$(u, T)(x_1 + k_1 L_1, x_2 + k_2 L_2, x_3) = (u, T)(x_1, x_2, x_3) \qquad (3.10)$$

As for any $k_1, k_2 \in Z$.

2. The Dirichlet boundary condition:

$$u = 0,\ T = 0 \quad \left(\text{Or } \frac{\partial T}{\partial n} = 0\right); \qquad (3.11)$$

3. The Free boundary condition:

$$T = 0,\ \ u_n = 0 \quad \left(\text{Or}\ \frac{\partial u_r}{\partial n} = 0\right), \tag{3.12}$$

For which n and τ are the unit normal as well as tangent vectors on $\partial D \times [0,1]$ accordingly, and $u_n = u \cdot n, u_r = u \cdot \tau$.

In order to minimize, the research will continue with this set of boundary conditions, and the entire outcomes hold true even for all other combinations of boundary conditions.

$$\begin{cases} T = 0,\ \ u = 0 \qquad \text{at}\ x_3 = (0,1) \\ (u,T)(x_1 + k_1 L_1, x_2 + k_2 L_2, x_3\, t) = (u,T)(u,t) \end{cases} \tag{3.13}$$

As for any $k_1, k_2 \in Z$.

4. Dynamic Bifurcation and Stability in the Rayleigh-Benard Convection

The significant milestone here is to try to demonstrate Rayleigh- Benard convention in terms of nonlinear theory by a new model of bifurcation, termed A_R (attractor bifurcation), and the respectful theories in the past studies [25]. all these are in line with the three features of a complete theory for the issue discussed along with the main idea with the techniques to employ.

4.1 The Dynamic bifurcation for the nonlinear progression equations:

As for this section, we have to apply some findings from the past study on dynamic bifurcation of abstract nonlinear evolution equations which had been suggested by many scholars particularly the one done by the authors in [26], which is required in this article for Benard problem. This section was structured to determine a formula or model for proving dynamic bifurcations for complications that involve symmetric linear operators.

4.2. Attractor bifurcation

By assuming H and H1 to be 2 Hilbert spaces, then $H1 \rightarrow H$ to be a compressed and compact insertion[2]. This works used the resulting nonlinear evolution equations.

$$\frac{\delta y}{\delta x} = L_{\beta Y} + G(Y, \beta) \tag{4.1}$$

$$Y(0) = Y_0 \tag{4.2}$$

Where Y: $(0,\infty) \longrightarrow H$ and represent the unknown function, $\beta \in \mathbb{R}$ as the system parameter, then, $L_\beta: H_1 \rightarrow H$ are parameterized linear that completely continuous fields which constantly depending on $\lambda \in R^1$, that satisfy the following equations:

$L_\lambda = -A + B_\lambda$ is a sectorial operator,

$A : H_1 \rightarrow H$ a linear homeomorphism, (4.3)

$B_\lambda : H_1 \rightarrow H$ the parameterized linear compact operators.

It is convenient to notices in some past studies [27],; that L_β gives an analytic semi-group as $\{e^{tL\beta}\}t \geq 0$. so, is possible to define fractional power operators $L_\beta^\propto$ for any $0 \leq \propto \leq 1$ having its domain $H_\propto = D(L_\beta^\propto)$ in such that $H_{\propto 1} \subset H_{\propto 2}$ if $\alpha 1 > \alpha_2$, and $H_0 = H$.

likewise, this research will consider that the nonlinear terms $G(y, \beta)$: $H_\propto \rightarrow H$ for some $1 > \alpha \geq 0$ are belong to the family of parameterized C^r bounded operator ($r \geq 1$) continuously depend on the parameter $\lambda \in R_1$, for which

$$G(y, \beta) = 0(\|y\|H_\propto) \quad (4.4)$$

In this illustrations, this research have what to do with the sectorial operator $L_\beta = -A + B_\beta$ for which a real eigenvalue sequence there exist $\{\rho k\} \subset R_1$ and an eigenvector

sequence of $\{e_k\} \subset H_1$ of A:

$$A_{ek} = \rho k^e k$$
$$0 < \rho_1 < \rho_2 < \cdots \quad (4.5)$$
$$\rho k \rightarrow \infty \, (k \rightarrow \infty)$$

for which $\{e_k\}$ is an orthogonal basis of H.

So, as for the compact operator $B_\beta : H_1 \rightarrow H$, this work will still presume that there will be a constant $0 < \theta < 1$ in such that

$B_\beta: H_\theta \rightarrow H$ Bounded, $\forall \lambda \in R^1$ (4.6)

Let this be $\{S_\beta(t)\}t \geq 0$ an operator semi-group formed by the equation (4.1) that delight in the properties.

For any $t \geq 0$, $S_\beta(t): H \rightarrow H$ is a linear continuous operator,

$S_\beta(0) = I : H \rightarrow H$ for the identity on H, and

Then, for any t, s ≥ 0, $S_\beta(t + s) = S_\beta(t) \cdot S_\beta(s)$

We can say, the solution of equation (4.1) and equation (4.2) can be articulated as

$$y(t) = S_\beta(t)y_0, \quad t \geq 0.$$

Definition 4.1. A set $\Sigma \subset H$ is called an invariant set of (4.1) if S (t) = Σ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (4.1) is said to be an attractor if Σ is compact, and there exists a neighborhood U⊂ H of Σ such that for any $\phi \in U$ we have

$\lim_{t\rightarrow\infty} \text{dist}_H(u\,(t,\varphi\,) = 0$ (4.7)

The largest open set U satisfying (4.7) is called the basin of attraction of Σ.

Definition 4.2.

We say that the equation (4.1) bifurcates from $(u,\lambda) = (0, \lambda_0)$ an invariant set Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (4.1), $0 \notin \Omega_{\lambda_n}$ such that

$\lim_{t\to\infty}\lambda_n = \lambda_0$

Lim max |x| = 0.

$n\to\infty$ $x\in\Omega_{\lambda_n}$

If the invariant sets Ω_λ are attractors of (4.1), then the bifurcation is called attractor bifurcation.

1. If Ω_λ are attractors and are homotopy equivalent to an m –dimensional sphere S^m, then the bifurcation is called S^m–attractor bifurcation

invariant set Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (4.1), 0 ∈/ Ω_{λ_n} such that

$$L_{\lambda x} = \alpha_1 x - \alpha_{2\,y}$$

$$L_{\lambda y} = \alpha_2 x - \alpha_{1\,y}$$

Now let the eigenvalues (counting the multiplicity) of L_λ be given by

$\beta_1(\lambda), \beta_{12}(\lambda), \cdots, \beta_K(\lambda) \in C$,

where C is the complex space. Suppose that

$$R_e\beta_i = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases} \qquad (1\leq i \leq m+1) \qquad (4.8)$$

$$R_e\beta_i(\lambda_0) < 0, \qquad \forall m+2 \leq j \qquad (4.9)$$

Let the eigenspace of L_λ at L_0be

$E_0 = U_1 \leq i \leq m+1\{u\epsilon H_1 \big| (L_{\lambda 0} - \beta_i(\lambda_0))^k\}$ u =0, K =1.2……}

It is known that dim E_0= m +1. The following dynamic bifurcation theorems for the (4.1) were proved in [28].

Theorem 4.3 (Attractor Bifurcation, [28]). Assume that the conditions (4.3), (4.4), (4.8) and (4.9) hold true, and u =0 is a locally asymptotically stable equilibrium point of (4.1) at $\lambda - \lambda_0$ Then the following assertions hold true.

1. (4.1) bifurcates from $(u,\lambda) = (0,\lambda_0)$ an attractor A_λ for $\lambda>\lambda_0$, with m ≤dim $A_\lambda \leq$ m +1, which is
connected as m >0;

2. the attractor $A_\lambda$is a limit of a sequence of (m +1)–dimensional annulus M_k with $M_k+1 \subset M_k$; especially ifA_λ is a finite simplicial complex, then A_λ has the homotopy type of S^m;

3.For any $u_\lambda \in A_\lambda$, u_λ can be expressed as

$$u_\lambda = v_\lambda + o(\|v_\lambda\|H_1), \qquad v_\lambda \in E_0:$$

2. If $G : H_1 \to H$ is compact, and the equilibrium points of (4.1) in A_λ are finite, then we have the index formula

$$\sum_{u_i \in A_\lambda} \mathrm{Ind}[-(L_\lambda + G), u_i] = \begin{cases} 2, & \text{if } m = \text{odd} \\ 0, & \text{if } m = \text{even} \end{cases}$$

5. If u =0 is globally stable for (4.1) at $\lambda = \lambda_0$, then for any bounded open set $u \subset H$ with $0 \in u$ there is an $\varepsilon>0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor A_λ bifurcated from $(0, \lambda_0)$ attracts U/ T in H, where T is the stable manifold of u =0 with co-dimension m +1. In particular, if (4.1) has global attractor for all λ near λ_0, then the ε here can be chosen independently of u

5. Attractor bifurcation of the B´enard problem:

5.1. Main theorems

The linearized equations of (3.4)-(3.6) are given by

$$, \quad \begin{cases} -\Delta u + \nabla p - \sqrt{R}TK = 0 \\ -\Delta T - \sqrt{Ru_3} = 0, \\ \mathrm{div} = 0, \end{cases} \tag{5.1}$$

where R is the Rayleigh number. These equations are supplemented with the same boundary conditions (3.13) as the nonlinear Boussinesq system[2]. This eigenvalue problem for the Rayleigh number R is symmetric. Hence[27], we know that all eigenvalues R_k with multiplicities m_k of (5.1) with (3.13) are real numbers, and

$$0 < R_1 < \cdots < R_k < R_k+1 < \cdots . \tag{5.2}$$

The first eigenvalue R_1, also denoted by $R_C = R_1$, is called the critical Rayleigh number. Let the multiplicity of $R_C$be m 1 = m +1 (m ≥ 0), and the first eigenvectors $\Psi_1 = (e_1\ (x), T_1), \cdots, \Psi_{m+1} = (e_{m+1}, T_{m+1})$ of (5.1) be orthonormal:

$$.\langle \psi_i, \psi_j \rangle_H = \int_\Omega \left[e_i \cdot e_j + T_i T_j \right] dx = \delta_{ij}$$

For simplicity, let E_0. be the first eigenspace of (5.1) with with (3.13)

$$E_0 = \left\{ \sum_{K=1}^{M+1} \alpha_k \psi_k \ \middle|\ \alpha_k \in \mathbb{R}, 1 \le k \le m+1 \right\} \tag{5.3}$$

The main results in this section are the following theorems.

Theorem 5.1. For the B´enard problem (3.4-3.6) with (3.13), the following assertions hold true.

1. When the Rayleigh number is less than or equal to the critical Rayleigh number: $R \leq R_C$ the steady state (u) = 0 is a globally asymptotically stable equilibrium point of the equations.
2. The equations bifurcate from ((u, T), R) = (0, R_C) an attractorA_R for $R > R_C$, with $m \leq \dim A_R \leq m + 1$, which is connected when m > 0.
3. For any (u , T) $\in A_R$, the velocity field u can be expressed as

$$u = \sum_{k=1}^{m+1} \alpha_k \, e_k + 0 \left(\sum_{k=1}^{m+1} \alpha_k \, e_k\right) \qquad (5.4)$$

where e_k are the velocity fields of the first eigenvectors in E_0.

4. The attractor A_R has the homotopy type of an m -dimensional sphere S^m provided A_R is a finite simplicial complex.
5. There are an open neighborhood u ⊂ Hof (u, T) = 0 and an ε > 0 such that as $R_C < R < R_C + \varepsilon$, the attractor A_R attracts u / T in H, where T is the stable manifold of (u, T) = 0 with co-dimension m + 1.

6. Remarks on topological structure of solutions of the Rayleigh-B´enard problem

As we mentioned before, the structure of the eigenvectors of the linearized problem (5.1) plays an important role for studying the onset of the Rayleigh-B´enardconvection. The dimension m +1 of the eigenspace E_0 determines the dimension of the bifurcated attractor A_R as well[1]. Hence in this section we examine in detail the first eigenspace for different geometry of the spatial domain and for different boundary conditions.

6.1. Solutions of the eigenvalue problem.

Hereafter, we always consider the B´enard problem on the rectangular region: $\Omega = (0, L_1) \times (0, L_2) \times (0,1)$, and the boundary condition taken as the free boundary condition

$$,u.n = 0 , \qquad \frac{\partial u.t}{\partial n} = 0 \text{ on } \partial\Omega \qquad (6.1)$$

$$T = 0 \quad \text{at} \quad x_3 = 0,1, \qquad (6.2)$$

$$\frac{dT}{dn} = 0 \quad \text{at} \quad x_1 = 0, L_1 \quad \text{or} \quad x_2 = 0, L_2 \qquad (6.3)$$

For the eigenvalue equations (5.1) with the boundary condition (6.1)—(6.3), we take the separation of variables as follows

$$,\begin{cases} (u_1.u_2) = \frac{1}{a^2}\left(\frac{\partial f(x_1.x_2)}{\partial x_1}, \frac{\partial f(x_1.x_2)}{\partial x_2}\right)\frac{dH(X_3)}{dx_3} \\ u_3 = f(x_1, x_2)H(x_3) \\ T = f(x_1, x_2)\alpha(x_3) \end{cases} \qquad (6.4)$$

where $a^2 > 0$ is an arbitrary constant.

It follows from (5.1) with (6.1)—(6.3) that the functions f, H, α satisfy

$$\begin{cases} -\Delta_1 f = a^2 f, \\ \frac{\partial f}{\partial x_1} = 0 \quad \text{at } x_1 = 0, L_1 \\ \frac{\partial f}{\partial x_2} = 0 \quad \text{at } x_2 = 0, L_2 \end{cases} \tag{6.5}$$

And

$$\begin{cases} \left(\frac{d^2}{dz^2} - a^2\right)^2 H = a^2 \lambda\alpha, \\ \left(\frac{d^2}{dz^2} - a^2\right)\alpha = -\lambda H, \end{cases} \tag{6.6}$$

supplemented with the boundary conditions

$$\begin{cases} \varphi(0) = \varphi(1) = 0\,, \\ H(0) = H(1) = 0\,, H''(0) = H''(1) = 0 \end{cases}$$

(6.7)

.

It is clear that the solutions of (6.5) are given by

$$\begin{cases} f(x_1, x_2) = \cos(a_1 x_1)\cos(a_2 x_2) \\ a_1^2 + a_2^2 = a^2,\ (a_1, a_2) = k_1\pi/L_1, k_2\pi/L_2 \end{cases} \tag{6.8}$$

,

for any $k_1, k_2 = 0,1 \dots ..$

Let $a_1^2 + a_2^2$=a^2.It is easy to see that for each given a^2, the first eigenvalue λ_0(a) and the eigenvectors of (6.6) and (6.7) are given by

$$\begin{cases} \lambda_0(a) = \frac{(\pi^2+a^2)^{3/2}}{a} \\ (H, \alpha) = (\sin\ \pi x_3, \frac{1}{a}\sqrt{\pi^2 + a^2}\sin \pi x_3 \end{cases} \tag{6.9}$$

.

It is easy to see that the first eigenvalue λ_1= $\sqrt{R_C}$ of (5.1) with (6.1)—(6.3) is the minimum of λ_0 (a):

$$R_C = \min_{a^2=a_1^2+a_2^2} \lambda_0^2(a)$$
$$= \min_{k_1,k_2 \in \mathbb{Z}} \left[\pi^4 + \left(1\frac{k_1^2}{L_1^2} + \frac{K_1^2}{L_2^2} \right)^3 / \left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} \right) \right] \quad (6.10)$$

Thus the first eigenvectors of (5.1) with (6.1)—(6.3) can be directly derived from (6.4), (6.8) and (6.9):

$$, \begin{cases} u_1 = -\frac{a_1\pi}{a^2} \sin(a_1x_1)\cos(a_2x_2)\cos(\pi x_3) \\ u_2 = -\frac{a_2\pi}{a^2} \cos(a_1x_1)\sin(a_2x_2)\cos(\pi x_3) \\ u_3 = \cos(a_1x_1)\cos(a_2x_2)\sin(\pi x_3) \\ T = -\frac{1}{a}\sqrt{\pi^2+a^2}\cos(a_1x_1)\cos(a_1x_2)\sin(\pi x_3) \end{cases} \quad (6.11).$$

where $a^2 = a_1^2 + a_2^2$ satisfies (6.10)

By Theorem 5.1, the topological structure of the bifurcated solutions of the B´enard problem (3.4–3.6) with (6.1)—(6.3) is determined by that of (6.11), and which depends, by (6.10), on the horizontal length scales $L_1$and L_2. Namely, the pattern of convection in the B´enard problem depends on the size and form of the containers of fluid[1]. This will be illustrated in the remaining part of this section.

6.2. Roll structure.

By (6.10) and (6.11) we know that when the length scales $L_1$and L_2 are given, the wave numbers k_1 and k_2 are derived, and the structure of the eigenvectors u of (5.1) are determined. Consider the case where

$$L_1 = L_2 = L, \quad \text{and} \quad 0 < L^2 < \frac{2-2^{1/3}}{2^{1/3}-1} \simeq 3 \quad (6.12)$$

We remark here that $L = hL_1/h$ is the aspect ratio between the horizontal scale and the vertical scale of the domain. In this case, the wave numbers (k_1, K_2) are given by

$(k_1 k_2)$ =(1,0) and (0,1),

and the eigenspace E_0 defined by (5.3) for the linearized Bousinesq equation (5.1) with boundary conditions (6.1-6.3) is two-dimensional and is given by

$$E_0 = \{\alpha_1\psi_1 + \alpha_2\psi_2 \mid \alpha_1\alpha_2 \in R\},$$

Where

$$\psi_I = (e_i T_i) \qquad i = 1,2$$

$$e_1 = \left(0, -L\sin\left(\frac{\pi x_2}{L}\right)\cos(\pi x_3), \cos\left(\frac{\pi x_2}{L}\right)\sin(\pi x_3)\right)$$

$$e_2 = \left(0, -L\sin\left(\frac{\pi x_2}{L}\right)\cos(\pi x_3), \cos\left(\frac{\pi x_2}{L}\right)\sin(\pi x_3)\right)$$

$$T_1 = \sqrt{L^2+1}\cos\left(\frac{\pi x_1}{L}\right)\sin(\pi x_3)$$

$T_2 = \sqrt{L^2 + 1}\cos\left(\frac{\pi x_2}{L}\right)\sin(\pi x_3)$

When $\alpha_1, \alpha_2 \neq 0$, the structure of $\varphi = \alpha_1\psi^1 + \alpha_2\psi^2 \in E_0$ is given schematically by Figure 6.1 (a)-(d).

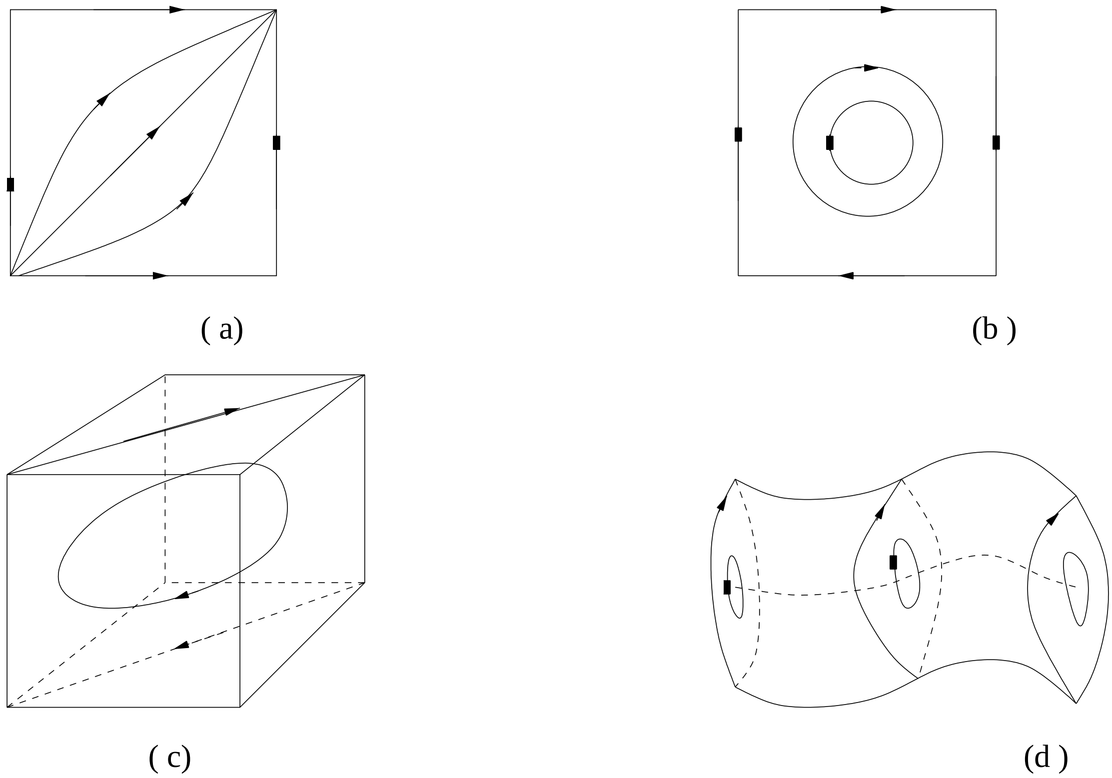

Fig. 6.1. Roll structure: (a) Flow structure on Z = 1, (b) flow structure on x = 1 or y = 0, (c) an elevation of the flow, and (d) flow structure in the interior of the cube.

The roll structure of $\varphi = \alpha_1\psi^1 + \alpha_2\psi^2 \in E_0$ has a certain stability, although it is not the structural stability, i.e. under a perturbation the roll trait remains invariant; we shall report on this new stability elsewhere.

Furthermore, the critical Rayleigh number is

$$R_c = \frac{\pi^4(1+L^2)^3}{L^4} \qquad (6.13)$$

By Theorem 5.1, we have the following results.

1. When the Rayleigh number $R \leq R_C$, the trivial solution $\emptyset = 0$ is globally asymptotically stable in H;
2. When the Rayleigh number $R_C, < R < R_C, + \varepsilon$ for some $\varepsilon > 0$, or when the temperature gradient satisfies

$$\frac{k\nu}{g\alpha}\frac{\pi^4(1+L^2)^3}{(Lh)^4} < \beta = \frac{T_0-T_1}{h} < \frac{k\nu}{g\alpha}\frac{\pi^4(1+L^2)^3}{(Lh)^4} + \varepsilon_1 \qquad (6.14)$$

the B´enard problem bifurcates from the trivial state $\emptyset = 0$ an attractor A_R with $1 \le \dim A_R \le 2$.

3. All solutions in A_R are small perturbations of the eigenvectors in E_0, having the roll structure.
4. As an attractor, A_R attracts $H-T$, where $T \subset H$ is a co-dimension 2 manifold. Hence, A_R is stable in the Lyapunov sense. Consequently, for any initial value $\varphi_0 \in H - T$, the solution $S_R(t)\varphi_0$ of the Boussinesq equations with (6.1) — (6.3) converges to A_R, which approximates the roll structure.

Remark 6.1. Since the eigenvector eigenspace E_0 has dimension two, the bifurcated attractor A_R has the homotopy type of cycles$_1$. In fact, it is possible that the bifurcated attractor is S_1. Since the spaces $E_1 = \{(u,\theta) \in H_1 \mid u_1 = 0\}$ and $E_2 = \{(u,\theta) \in H_1 \mid u_2 = 0\}$ is invariant for the equation (5.1), the bifurcated attractor Σ contains at least four singular points. If $\Sigma = s_1$, then Σ has exactly four singular points, and two of which are the minimal attractors see ; [1].

7. Conclusion

In this article we tried as we mentioned before to clarify the structure of the eigenvectors of the linearized problem in which this study plays an important role and studying the onset of the Rayleigh-B´enard convection. The dimension $m+1$ of the eingenspace E_0 regulates the dimension of the bifurcated attractor A_R as well. However, the thesis will further explain by examine the detail of the first eingenspace for different geometry of the spatial domain and for different geometry and boundary condions.

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